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METHOD AND BUSINESS PROCESS FOR THE ESTIMATION OF MEAN PRODUCTION FOR ASSEMBLE-TO-ORDER MANUFACTURING OPERATIONS

Paul Dagum, Michael Goldbach and Balazs Kralik

Related Applications

This application is a continuation of US Provisional Application Serial Number 60/213,189 filed June 21, 2000. That disclosure is incorporated in this document by reference.

Technical Field of the Invention

The present invention is generally related to resource-planning methods used by manufacturing companies and other organizations. More specifically, the present invention includes a strategic resource planning method that accounts for uncertainties inherent in the forecasting process.

Background of the Invention

In the field of product manufacturing, the term assemble-to-order (ATO) refers to systems where products are not produced until the demand for the products becomes known. Manufacture-to-stock (MTS) systems, on the other hand, base production on component availability. Unlike ATO manufacturing, MTS systems often produce products before the demand for those products has been determined.

Manufacturing companies typically prefer to use ATO manufacturing operations whenever possible. ATO allows manufacturers to minimize their component inventories. This

reduces the amount of capital invested in components and reduces the risk that components will loose value or become obsolete before they are transformed and sold as products.

In ATO manufacturing operations, it is paramount for decision-makers to be able to accurately forecast production on a period-by-period basis. The absence of accurate forecasting means that makes the estimation of important quantities such as expected revenues and costs difficult and in some cases impossible.

Unfortunately, accurately forecasting production can be problematic. Product demand is necessarily an important factor in estimations of this type. At the same time, product demand is not, by itself, a sufficient basis for forecasting. This follows because production can (and often is) constrained by the availability of the components needed to assemble the products. As a result, there is always a chance that the demand for a product will not be met within a particular period. Typically, this means that customers are either turned away or that orders are delayed – the product is "back ordered."

To accurately forecast production, manufactures must be able to compute expected production from stochastic demand data, component consumption data and component levels. In many cases, this computation will involve a plurality of product and component kinds, potentially numbering in the thousands or more. Figure 1 describes this problem for a simplified case where there are two products P1 and P2 and two components C1 and C2.

In Figure 1, the production constraints imposed by the components C1 and C2 are shown as two dashed lines labeled C1 and C2. These lines divide the space of possible production values for the pair (P1, P2) into feasible and infeasible regions. The feasible region is the portion of the positive quadrant under the lines C1 and C2.

The probability distribution of demand is illustrated by the concentric ellipses. Each ellipse is an iso-probability curve (i.e. the line connects (P1, P2) points of equal probability). It is clear that there is a nonzero chance that the demand for the product pair may fall in the infeasible region.

If demand falls in the infeasible region, actual production will be less than demand. Hence mean production will be less than mean demand. There is a need for systems that can

accurately calculate the differences between mean demand and mean production. This is particularly true for manufacturing operations that involve large numbers of products or large numbers of components. It is also particularly true where the markets for products or the market for components are volatile.

Summary of the Invention

An embodiment of the present invention includes a method for forecasting the mean production (the expected production) for a target planning period. To begin this method, a user, or planner chooses one or more products for which the expected production is desired.

The user then enters data describing each selected product. The data entered for each product includes data describing the demand for that product as well as data describing the components required for each product. Inter-product dependencies are also entered.

The expected production for each selected product for the target planning-period is expressed as a sum of multidimensional integrals involving the data entered in the previous two steps. Once formulated, the integrals are evaluated. The result of this computation is then presented to the user.

Stated differently, the present invention includes a method for the estimation of mean production for assemble-to-order manufacturing operations, the method comprising the steps of: specifying a product to be analyzed; entering data describing the components required to produce the specified product; formulating a sum of multidimensional integrals corresponding to the estimation of mean production for the specified product; and evaluating the sum of multidimensional integrals.

Other aspects and advantages of the present invention will become apparent from the following descriptions and accompanying drawings.

Brief Description of the Drawings

For a more complete understanding of the present invention and for further features and advantages, reference is now made to the following description taken in conjunction with the accompanying drawings, in which:

Figure 1 is a graph showing production as it relates to demand in both feasible and infeasible regions.

Figure 2 is a block diagram of an Internet-like network shown as a representative environment for deployment of the present invention.

Figure 3 is a block diagram of a computer system as used within the network of Figure 1.

Figure 4 is a flowchart showing the steps associated with an embodiment of the mean production forecasting method of the present invention.

Figure 5 is a flowchart showing the sub-steps associated with the step of entering data about products and components as used within the method of Figure 4.

Figure 6 is a flowchart showing the sub-steps associated with the step of generating a sum of integrals as used within the method of Figure 4.

Figure 7 is a flowchart showing the sub-steps associated with the step of selecting a production policy as used within the method of Figure 4.

Detailed Description of the Preferred Embodiments

The preferred embodiments of the present invention and their advantages are best understood by referring to Figures 1 through 4 of the drawings. Like numerals are used for like and corresponding parts of the various drawings.

Definitions

Component plan: a list of quantities for each component, representing a company's component order for a given planning period.

Scenario: a set of assumptions about products and components. A scenario includes product parameters, component parameters, component consumption, component interactions, and an allocation policy.

Environment

In Figure 2, a computer network 200 is shown as a representative environment for an embodiment of the present invention. Computer network 200 is intended to be representative of the complete spectrum of computer network types including Internet and Internet-like networks. Computer network 200 includes a number of computer systems, of which computer system 202a through 202f are representative. Computer systems 202 are intended to be representative of the wide range of large and small computer and computer-like devices that are used in computer networks of all types. Computer systems 202 are specifically intended to include non-traditional computing devices such as personal digital assistants and web-enabled cellular telephones.

Figure 3 shows a representative implementation for computer systems 202. Structurally, each computer system 202 includes a processor, or processors 300, and a memory 302. Processor 300 can be selected from a wide range of commercially available or custom types. An input device 304 and an output device 306 are connected to processor 300 and memory 302. Input device 304 and output device 306 represent all types of I/O devices such as disk drives, keyboards, modems, network adapters, printers and displays. Each computer system 202 may also includes a disk drive 310 of any suitable disk drive type (equivalently, disk drive 310 may be any non-volatile mass storage system such as "flash" memory).

Overview of Method and Apparatus for Resource Plan Analysis Under Uncertainty

As shown in Figure 4, an embodiment of the present invention includes a Method 400 for forecasting the mean production (the expected production) for a target planning period. Method 400 begins with step 402 where a user, or planner, chooses one or more products for which the expected production is desired.

In step 404 the user enters data describing each product selected in step 402. The data entered for each product includes data describing the demand for that product as well as data describing the components required for each product. Inter-product dependencies are also entered.

In step 406 the expected production for the target planning-period is expressed as a sum of multidimensional integrals involving the data entered in the previous two steps.

In step 408 the sum of integrals formulated in the previous step is evaluated. This evaluation may be performed using a range of different methods including Monte Carlo simulation and quadratures. It is preferable, however to use the method disclosed in the copending commonly owned application "Method And Apparatus For Multivariate Allocation Of Resources." This method is particularly preferable in cases when Method 400 is being performed for large numbers of products or components. In these cases, other evaluation methods may be too slow or unstable.

In step 410 the result of the computation of step 408 is presented to the user.

Product Selection

In step 402, the user identifies the m products that will be analyzed.

Product and Component Data Specification

Step 404, where the user specifies the required data for the mean production calculation, can be further subdivided into the sequence of steps shown in Figure 5. The initial portion of this sequence is a loop that includes steps 404-2 through 404-10. In this loop, data is input for each of the m products that have been selected for processing by Method 400. Data is also input for each of the n components that are required (either directly or indirectly) to produce the products that have been selected for processing by Method 400

In the loop of steps 404-2 through 404-10 the following information is entered:

1) The type of components required to produce each product,

1) The number of each component required to produce each product (**A**),

1) The allocation of each component (**d**),

1) The mean demand for each product (μ),

1) The standard deviation of the demand for each product,

1) The correlation between each product and any other product.

Within this loop, each data item can be input manually by the user (see step 402-6) or retrieved from a user database (see step 402-8).

In the loop of steps 404-2 through 404-10, an m x n matrix **A** is created. The matrix **A** includes one column $\mathbf{a}_i$ for each of the m products selected for analysis. Each column $\mathbf{a}_i$ contains the bill of materials required to produce one product. Each of the n elements of a given $\mathbf{a}_i$ specifies the quantity of a particular component required to produce the corresponding product.

The loop of steps 404-2 through 404-10 also generates a vector **d**. The vector **d** includes a total of n elements, one for each component. Each element within **d** specifies the available quantity of a corresponding component.

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The loop of steps 404-2 through 404-10 also generates a vector $\boldsymbol{\mu}$. The vector $\boldsymbol{\mu}$ includes a total of m elements, one for each product. Each element within $\boldsymbol{\mu}$ specifies the mean demand for a corresponding product.

Once the data required for each product and component has been input, processing continues at step 404-12. In step 404-12, the product correlations (item 6 above) and demand standard deviations (item 4 above) input in the preceding loop are combined to create a covariance matrix Σ.

<u>Formulation of Expected Production as a Sum of Multidimensional Integrals</u>

In Step 406, the covariance matrix is used to formulate a sum of multidimensional integrals. This step (i.e., creation of the multidimensional integrals) can be further subdivided into the sequence of steps shown in Figure 6.

In step 406-2, a feasible region (denoted Ω) is defined. The feasible region Ω is defined to include all points where the demand for a product can be met with the current levels of the

components needed to manufacture the product. To determine Ω, the matrix $\mathbf{A}$ and vector $\mathbf{d}$ generated in step 404 are used. Ω is defined as the space of those $\mathbf{q}$ vectors, for which the inequality $\mathbf{d} - \mathbf{A}\mathbf{q} > 0$ holds, where $\mathbf{q}$ is a vector variable denoting the production of the products.

In step 406-4 a production policy is specified for the feasible region Ω. The production policy for a region is a function that relates production to demand. Within Ω, component availability $\mathbf{d}$ equals or exceeds demand. As a result, production is equivalent to demand within the feasible region Ω. For this reason, the production policy for Ω is set to $\mathbf{q}(\mathbf{x}) = \mathbf{x}$ reflecting the fact that production is capable of mirroring demand within Ω.

In step 406-6, an infeasible region (denoted $\overline{\Omega}$) is determined. The infeasible region $\overline{\Omega}$ is the space of all vectors, where the demand for a product exceeds the supply of components needed to manufacture the product. $\overline{\Omega}$ includes all vectors, or equivalently points, which are not included in Ω. $\overline{\Omega}$ is defined as the complement of Ω as determined in step 406-2. In general, Ω and $\overline{\Omega}$ are separated by a hyperplane in the n dimensional space $\Re^n$.

In step 406-8 a production policy for the infeasible region $\overline{\Omega}$ is formulated. Unlike production in the feasible region Ω, production within the infeasible region $\overline{\Omega}$ does not mirror demand. This is because component demand exceeds component supply within $\overline{\Omega}$. In step 406-8 a production policy is specified to determine how production will proceed in light of component supply inadequacies. The production policy is a function $\mathbf{q}(\mathbf{x})$ that maps demand to production in $\overline{\Omega}$.

A range of different production policies may be used in step 406-8. For the particular implementation being described, a production policy known as the uniform production policy is used. The uniform production policy relates production $\mathbf{q}$ to demand $\mathbf{x}$ within the infeasible region $\overline{\Omega}$ using the function: $\mathbf{q}(\mathbf{x}) = \mathbf{x} - \alpha(\mathbf{x}) \mathbf{u}$. In this function, $\mathbf{u}$ is a production policy vector that the user may optionally specify. The production policy vector $\mathbf{u}$ shows the amounts by which demand for products has to be decreased until the demand is feasible. $\alpha(\mathbf{x})$ is defined as:

$$\alpha(\mathbf{x}) = \max_{i \in [m]} \left[\frac{\mathbf{a}_i \cdot \mathbf{x} - d_i}{\mathbf{a}_i \cdot \mathbf{u}} \right]$$

This formulation of $\alpha(\mathbf{x})$ ensures that, for any $\mathbf{x} \in \overline{\Omega}$ and a given $\mathbf{u}$, the production policy $\alpha(\mathbf{x})$ is the highest level such that $\mathbf{q}(\mathbf{x}) < \mathbf{x}$ and $\mathbf{d} - \mathbf{Aq} > 0$. The values of the elements of the constraint matrix **A** can be calculated from the components of the policy vector **u** and $\alpha(\mathbf{x})$.

In step 406-10 the expected or mean production $\overline{\mathbf{q}}$ is expressed as the following sum of multidimensional integrals:

$$\overline{\mathrm{q}} = \int_{\Omega} \mathrm{x} \;\; f(\mathrm{x}, \;\; \boldsymbol{\mu}, \;\; \Sigma) \;\; d\mathrm{x} \;\; + \;\; \int_{\overline{\Omega}} \mathrm{q}(\mathrm{x}) \;\; f(\mathrm{x}, \;\; \boldsymbol{\mu}, \;\; \Sigma) \;\; d\mathrm{x}$$

where $f(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma})$ is the multivariate normal density function with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.

Evaluation of Multidimensional Integrals

In step 408 the sum of the integrals, which were formulated in the previous step, is evaluated to compute the expected or mean production $\overline{\mathbf{q}}$. To compute $\overline{\mathbf{q}}$, $\alpha(\mathbf{x})$ is rewritten using the Cholesky decomposition of Σ. As previously noted, Σ is the covariance matrix originally computed using the product demand standard distributions and inter-product correlations (see step 404-12). The Cholesky decomposition of the covariance matrix Σ can be written as **TT'**=Σ, where **T** is a lower triangular matrix. A benefit of the Cholesky decomposition is that the transformed random variables **z** have zero mean and I_n, the identity matrix as their covariance matrix. With this transformation of variables **x** may be rewritten as $\mathbf{x} = T\mathbf{z} + \boldsymbol{\mu}$, and the feasible region Ω also gets transformed into Ω'.

The Case of Single Component Constraints

In this section we demonstrate the essentials of the technique by first describing the case, when there is a constraint only for a single component. In this case the feasible region Ω is

defined by the constraint equation $d - \mathbf{a}\cdot\boldsymbol{\mu} - \mathbf{b}\cdot\mathbf{z} = 0$, where $\mathbf{b} = \mathbf{T'a}$, and d is a scalar number, describing the single constraint.

After the Cholesky transformation the next step is to perform an orthogonal rotation **R**, which transforms the vector **b** to point along a unit vector of the new system, say $\mathbf{e_1}$: $\mathbf{Rb} = |\mathbf{b}|\, \mathbf{e}_1$, where $|\mathbf{b}|$ is the magnitude of vector **b**. After the rotation the new variables will be denoted by **w**: $\mathbf{w} = \mathbf{Rz}$. Because of the orthonormality of **R** the covariance matrix of **w** will still be the identity matrix. An advantage of this rotation is that the hyperplane, defining the feasible region Ω is now perpendicular to only one axis, defined by $\mathbf{e}_1$, and it is parallel to all the other axis of the rotated coordinate system. Therefore the n dimensional integral, confined by a generic hyperplane, is transformed into an integral, which is confined along a single axis, and is unconfined in the other n-1 directions. The unconfined integrals are straightforward to execute and we are left with a singlevariate integral, which is much easier to perform. In mathematical form the integral of some function $h(\mathbf{x})$ over the feasible region can be written as:

$$I = \int_{\Re^{n-1}} f(\mathbf{w}_{n-1};0,I_{n-1})d\mathbf{w}_{n-1} \int_{-\infty}^{U} h''(\mathbf{w})\, f(w_1;0,1)\, dw_1$$

Here h''(w) is the generic function h(x) after the Cholesky transformation **T** and the rotation **R**, $\mathbf{w}_{n-1} = (w_2, \ldots, w_n)$, and $\Re^{n-1}$ is the (n-1) dimensional space of real numbers. Finally $U = (d - \mathbf{a}\boldsymbol{\mu}) / |\,\mathbf{b}\,|$, identifying the intersection of the hyperplane and the $\mathbf{e}_1$ axis.

In the special case of $h(\mathbf{x}) = \mathrm{d} - \mathbf{ux}$, the twice transformed function takes the form :

$$h''(\mathbf{w}) = d - \mathbf{u}\cdot\boldsymbol{\mu} - \mathbf{RT'u}\cdot\mathbf{w} \quad .$$

With this the integral I can be rewritten as

$$I = (d - \mathbf{u}\cdot\boldsymbol{\mu})\, \Im[-\infty, U] - s_1\, \Im_x[-\infty, U] \quad .$$

Here

$$\Im[-\infty, U] = \int_{\Re^{n-1}} f(\mathbf{w}_{n-1};0,I_{n-1})d\mathbf{w}_{n-1} \int_{-\infty}^{U} f(w_1;0,1)\, dw_1 \quad ,$$

and

$$\Im_x[-\infty,U] = \int_{\Re^{n-1}} f(\mathbf{w}_{n-1};0,I_{n-1})d\mathbf{w}_{n-1} \int_{-\infty}^{U} w_1 f(w_1;0,1)\, dw_1$$

finally

$$s_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} R_{1i}T_{ij}'u_j$$

To summarize: as a result of the Cholesky transformation and the rotation **R,** the original I multivariate integral over a confined subspace has been reduced to a univariate definite integral, which can be executed often analytically, and always numerically.

The Case of Multicomponent Constraints

We now consider the case of having several constraints for the availability of components. These constraints can be written as a set of constraint equations

$$\mathbf{d} - \mathbf{A}\mathbf{q} > 0$$

where **d** is a p dimensional vector, and **A** is a p times n dimensional matrix. Here p is less than or equal to n. The i-th row of the matrix **A** corresponds to the constraint vector $\mathbf{a}_i$ similarly to the case of the single component constraint. The first step of the method, the Cholesky transformation, can be performed in the present case as well, transforming to the new variables **z.** The constraint vectors $\mathbf{a}_i$ are transformed into the vectors $\mathbf{b}_i$ by this transformation.

These equations once again define the feasible region, which is now bounded by a set of hyperplanes. However in a generic case the hyperplanes are not orthogonal to each other. For this reason in general it is hard to separate the multivariate integrals into univariate integrals.

An approximate evaluation of the integral I can be constructed the following way. Define a new r-dimensional orthogonal basis **u** defined by a set of r hyperplanes, such that each of the original m hyperplanes is mapped to a new hyperplane. In general r is less than or equal to m and in practice r is often 1. The index mapping from the original m-dimensional basis to the new r-dimensional set is represented by σ such that $\sigma(i)$ is the index of new hyperplane associated

with the original hyperplane i.. The new basis may be defined in various ways with the objective being to closely approximate the originalfeasible region. In practice we take the new basis to be a single orthanormally transformed hyperplane.

In general the dimension of the space of these new **u** orthogonal constraint vectors can be smaller than those of the original **b** constraint vectors. Therefore some of the transformed hyperplanes, which are now orthogonal to only one axis, will be parallel to each other. In this case the integration along the say j-th axis should go up to c_j, the minimum of the integrational limits along these axis. In terms of the original variables:

$$c_j = \min_{i \in \sigma^{-1}(j)} [\frac{d_i - \mathbf{a}_i \boldsymbol{\mu}}{|\mathbf{b}_i|}]$$

With this finally the integral I is given as:

$$I = \int_{\Re^{n-r}} f(\mathbf{w}_{n-r};0,I_{n-r})d\mathbf{w}_{n-r} \prod_{j=1}^{r} \int_{-\infty}^{c_j} h''(\mathbf{w})\ f(w_j;0,1)\ dw_j$$

where $r < m$.

Now we will employ the above derived relations for the case, when we have to integrate a linear function of the variable $h(\mathbf{x})=\mathrm{d}_i-\mathbf{a}_i\mathbf{x}$. The twice-transformed function takes the form:

$$h''(\mathbf{w}) = \mathrm{d}_i - \mathbf{a}_i \cdot \boldsymbol{\mu} - \mathbf{RT'a}_i \cdot \mathbf{w} \quad .$$

Introducing $\mathbf{s} = \mathbf{RT'a}_i$

With this the integral I can be rewritten as

$$I = (\mathrm{d}_i - \mathbf{a}_i \cdot \boldsymbol{\mu}) \prod_{i=1}^{r} \Im[-\infty, c_i] - \sum_{j=1}^{r} s_j\ \Im_x[-\infty, c_j] \prod_{i=1,i\neq j}^{r} \Im[-\infty, c_i] .$$

$$s_j = \sum_{i=1}^{n}\sum_{j=1}^{n} R_{ji}T_{ik}'(a_i)_k \quad .$$

With all these preparations, we are now ready to evaluate the integral of the production policy.

We start with performing the same Cholesky decomposition, which allows $\alpha(\mathbf{x})$ to be reduced as:

$$\alpha'(\mathbf{z}) = \alpha(T\mathbf{z} + \boldsymbol{\mu}) = \max_{i \in [m]} \left[\frac{\mathbf{b}_i \cdot \mathbf{z} + \mathbf{a}_i \cdot \boldsymbol{\mu} - d_i}{\mathbf{a}_i \cdot \mathbf{u}} \right]$$

where again $\mathbf{b}_i = T'\mathbf{a}_i + \boldsymbol{\mu}$.

Next the same orthonormal rotation is performed as before, using the matrix **R**. Then the m vectors $\mathbf{b}_i$ are again approximated by r orthonormal vectors $\mathbf{u}_j$ such that $\mathbf{b}_i \cong |\mathbf{b}_i| \; \mathbf{u}_{\sigma(i)}$. This approximation identifies mutually orthogonal hyperplanes, which will allow for a more straightforward evaluation of the integral I. As before, σ is a mapping from the index set $[m]$ into $[r]$. Substituting for the $\mathbf{b}_i$ and defining the rotation matrix R such that $R\mathbf{u}_{\sigma(i)} = \mathbf{e}_{\sigma(i)}$ where $\mathbf{e}_i$ are unit vectors yields:

$$\alpha''(\mathbf{w}) = \alpha(R'\mathbf{w}) = \max_{i \in [m]} \left[\beta_i w_{\sigma(i)} + \gamma_i\right].$$

Noting that $|\mathbf{b}_i| = \sqrt{\mathbf{a}_i' \sum \mathbf{a}_i}$

$$\beta_i = \frac{\sqrt{\mathbf{a}_i' \sum \mathbf{a}_i}}{\mathbf{a}_i \cdot \mathbf{u}}$$

and $\gamma_i = \dfrac{\mathbf{a}_i \cdot \boldsymbol{\mu} - d_i}{\mathbf{a}_i \cdot \mathbf{u}}$

allows $\alpha''(\mathbf{w})$ to be rewritten as:

$$\alpha''(\mathbf{w}) = \max_{i \in [r]} \left[\max_{j \in \sigma^{-1}(i)} \left(\beta_j w_i + \gamma_j\right)\right].$$

We are now ready to compute the value of the expected production. An important part of the calculation is to evaluate the following integral

$$I^* = \int_{\Omega} \alpha(\mathbf{x}) \quad f(\mathbf{x}, \quad \boldsymbol{\mu}, \quad \boldsymbol{\Sigma}) \quad d\mathbf{x} \ .$$

Using the above preparatory formulae I* reduces to

$$I^* = \prod_{i=1}^{r} \int_{-\infty}^{c_i} \alpha''(\mathbf{w}) \quad f(\mathbf{w}, \quad 0, \quad I_n) \quad d\mathbf{w}$$

where we used that the fact that the f multivariate normal density function is normalized to one, thus in the directions where the integration is unconstrained, the integrals can be readily performed to yield unity. Hence only those integrals remain, where a hyperplane introduces a c_i upper bound for the integral:

$$c_i = \min_{j \in \sigma^{-1}(i)} \left[\frac{d_j - \mathbf{a}_j - \boldsymbol{\mu}}{|\mathbf{b}_j|} \right] \ .$$

Since $\alpha''(\mathbf{w})$ is a linear function of its argument, once again the preparatory formulae can be applied to yield

$$I^* = \sum_{i=1}^{r} \int_{-\infty}^{c_i} \max_{j \in \sigma^{-1}(i)} (\beta_j w_i + \gamma_j) \prod_{k \neq i} \Im[-\infty, \quad \min(c_k, \quad U_{ik}(x))] \quad f(\mathbf{x}, \quad 0, \quad 1) \quad dx$$

Using the notation

$$M_i = \max_{j \in \sigma^{-1}(i)} (\beta_j w_i + \gamma_j)$$

the following describes the integrational limit U_{ik}:

$$U_{ik} = \min_{j \in \sigma^{-1}(i)} \left[\frac{M_i - \gamma_j}{\beta_j} \right]$$

In the final analysis the original n dimensional multivariate integral with an arbitrary covariance matrix, constrained by a set of p arbitrary hyperplanes, has been reduced to r independent univariate definite integrals. Note that when r=1, $\sigma^{-1}(i)$ is the entire set $[m]$.

As can be shown, the original integral is an NP hard problem, requiring exponentially long time for evaluation, whereas the just-derived univariate integrals can be evaluated in a straightforward manner by any known numerical technique, including even the Simpson procedure.

The total expected production is then given by

$$\bar{\mathbf{q}} = \int_{\Omega} \mathbf{x} \; f(\mathbf{x}, \; \boldsymbol{\mu}, \; \Sigma) \; d\mathbf{x} + \int_{\bar{\Omega}} \mathbf{q}(\mathbf{x}) \; f(\mathbf{x}, \; \boldsymbol{\mu}, \; \Sigma) \; d\mathbf{x}$$

$$\bar{\mathbf{q}} = \int_{\Omega+\bar{\Omega}} \mathbf{x} \; f(\mathbf{x}, \; \boldsymbol{\mu}, \; \Sigma) \; d\mathbf{x} - \mathbf{u} \int_{\bar{\Omega}} \alpha(\mathbf{x}) \; f(\mathbf{x}, \; \boldsymbol{\mu}, \; \Sigma) \; d\mathbf{x}$$

$$\bar{\mathbf{q}} = \int_{\Omega+\bar{\Omega}} \mathbf{x} \; f(\mathbf{x}, \; \boldsymbol{\mu}, \; \Sigma) \; d\mathbf{x} - \mathbf{u} \int_{\Omega+\bar{\Omega}} \alpha(\mathbf{x}) \; f(\mathbf{x}, \; \boldsymbol{\mu}, \; \Sigma) \; d\mathbf{x} + \mathbf{u} \int_{\Omega} \alpha(\mathbf{x}) \; f(\mathbf{x}, \; \boldsymbol{\mu}, \; \Sigma) \; d\mathbf{x}$$

Here one recognizes that the integration over the $\Omega+\bar{\Omega}$ space is an integration over the total space, not constrained by the hyperplanes, and as such, can be readily calculated. Finally the last term is exactly what has been calculated above, thus the above result for I* can be directly employed here, yielding the desired expected production.

The Local u-Policy

The uniform policy prescribes a demand-production mapping that is manifestly irrational in some cases. In particular, the problem of "intercomponent" effects arises where a component that gates production for one product will diminish production over all. This arises because the previous policies compute a single α for all products.

On an event by event bases, an "iterated policy" represents rational demand-production mappings, but analytic formulas are not available for the expected production over these policies. A compromise is to compute a separate α for each product. This we refer to as the *local u-Policy*.

As before, for product *i*, let $D(i)$ denote the components in the BOM of *i*. Also, as before, for any component *j* we can define the subspaces $\Omega \;=\; \{\mathbf{x} : d_j - \mathbf{a}_j \cdot \mathbf{x} > 0\}$. We now define the subspaces Ω^i for each product *i* as

$$\Omega_i = \bigcap\nolimits_{j \in D(i)} \Omega_j$$

If $\mathbf{x} \in \Omega^i$ then component allocation is sufficient to meet product demand for the i^{th} product, even though it may not be sufficient to meet demand for other products.

We make the important observation that in computing expected values for functions $h(\mathbf{x})$ that can be separated into a sum,

$$h(\mathbf{x}) = h_1(x_1) + \ldots + h_n(x_n)$$

the appropriate subspace of integration for the *feasible* production for each function h_j is Ω^j.

For any subspace Ω, let $\mathbf{E}_\Omega$ denote the expectation over support Ω. Thus, the expectation of *h* can be expressed as

$$\mathbf{E}h = \sum_{i=1}^{n} \left| \mathbf{E}_{\Omega^i} h_i(x_i) + \mathbf{E}_{\bar{\Omega}^i} h_i(q(\mathbf{x})) \right|$$

where $\mathbf{q}(\mathbf{x})$ is any production policy that maps each $\mathbf{x} \in \bar{\Omega}^i$ into $q_i(\mathbf{x}) \in \Omega^i$.

We define the *local u*-policy $\mathbf{q}$ as

$$q_i(x) = \begin{cases} x_i & if\, x \in \Omega_i \\ x_i - \alpha_i(x,d)u_i & otherwise \end{cases}$$

where

$$\alpha_i(\mathbf{x},\mathbf{d}) = \max_{j \in D(i)} \left[\frac{\mathbf{a}_j \cdot \mathbf{x} - d_j}{\mathbf{a}_j \cdot \mathbf{u}} \right]$$

We now show that with the preceding definition, for any *k*, we have that $d_k - \mathbf{a}_k \cdot \mathbf{q} > 0$. Substituting $\mathbf{q}$ we get

$$d_k - \mathbf{a}_k \cdot q = d_k - \mathbf{a}_k \cdot x + \sum_{i=1}^{n} a_{ki} u_i \max_{j \in D(i)} \left[\frac{\mathbf{a}_j \cdot \mathbf{x} - d_j}{\mathbf{a}_j \cdot \mathbf{u}} \right]$$

But observe that $a_{ki} = 0$ if $k \notin D(i)$, and therefore

$$\max_{j \in D(i)} \left[\frac{\mathbf{a}_j \cdot \mathbf{x} - d_j}{\mathbf{a}_j \cdot \mathbf{u}} \right] \geq \frac{\mathbf{a}_k \cdot \mathbf{x} - d_k}{\mathbf{a}_k \cdot \mathbf{u}}$$

proving that $d_k - \mathbf{a}_k \cdot \mathbf{q} > 0$.

Computation of the results in this case is identical to the method described in the previous section except that a separate integration is done for each element of the expected production vector. For each element i, the index set of constraints [m] is replaced by the reduced set $D(i)$ which includes only those components in the BOM for product **i**. (i.e., the index of rows in the i-th column of the connect matrix A for which the entry is nonzero).

Although particular embodiments of the present invention have been shown and described, it will be obvious to those skilled in the art that changes and modifications may be made without departing from the present invention in its broader aspects, and therefore, the appended claims are to encompass within their scope all such changes and modifications that fall within the true scope of the present invention.

1 Expectation of General Functions Under a Multivariate Normal Distribution

We introduce some proprietary results on the properties of of the multivariate normal distribution over arbitrary half-spaces. Let $f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})$ denote the multivariate normal density function of $\mathbf{x} \in \Re^n$ with mean vector μ and covariance matrix Σ. Let Ω denote the half space $\Omega = \{\mathbf{x} : d - \mathbf{a} \cdot \mathbf{x} > 0\}$ and let $h(\mathbf{x})$ denote an arbitrary multivariate function of $\mathbf{x}$. We show how to obtain an expression for the following general integrals:

$$I_1 = \int_{\Omega} h(\mathbf{x}) f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) d\mathbf{x} \tag{1.1}$$

and the dual integral

$$I_2 = \int_{\bar{\Omega}} h(\mathbf{x}) f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) d\mathbf{x}, \tag{1.2}$$

where $\bar{\Omega} = \{\mathbf{x} : d - \mathbf{a} \cdot \mathbf{x} \leq 0\}$.

The solution to I_1 and I_2 can be obtained by performing two transformations. We first perform a Cholesky decomposition that transforms the multivariate normal distribution into a multivariate normal with zero mean and identity matrix for the covariance matrix. The Cholesky decomposition amounts to a linear transformation of the space $\Re^n$, and therefore transforms Ω into another halfspace Ω'. After the Cholesky decompositions, we rotate the space so that the hyperplane defining the halfspace Ω' lies along one of the coordinate axis in the Cholesky decomposition space. We exploit the fact that the zero-mean, identity matrix covariance matrix multivariate normal distribution is invariant under $SO(n)$ group transformations, where $SO(n)$ denotes the Lie group of orthogonal rotations in $\Re^n$.

1.1 Cholesky Decomposition

For a positive definite matrix Σ, the Cholesky decomposition defines a unique lower triangular matrix T such that $TT' = \Sigma$ [Tong, p 184]. We note that $\mathbf{x}$ has distribution $\mathcal{N}(\mu, \Sigma)$. Let T denote the matrix of the Cholesky decomposition of Σ. It follows from the linear transformation properties of multivariate normal distributions that the random variable $\mathbf{z} = T^{-1}(\mathbf{x}-\mu)$ has distribution $\mathcal{N}(0, I_n)$ [Tong, p 32]. With that transformation, the defining hyperplane for the subspace Ω becomes

$$d - \mathbf{a} \cdot (T\mathbf{z} + \mu) = 0,$$

or

$$d - \mathbf{a} \cdot \mu - \mathbf{b} \cdot \mathbf{z} = 0 \tag{1.3}$$

where $\mathbf{b} = T'\mathbf{a}$. The transformed hyperplane defines the new region $\Omega' = \{\mathbf{z} : d - \mathbf{a} \cdot \mu - \mathbf{b} \cdot \mathbf{z} > 0\}$. The transformation also defines a new function $h'(\mathbf{z}) = h(T\mathbf{z} + \mu)$. Thus, we get

$$I_1 = \int_{\Omega'} h'(\mathbf{z}) f(\mathbf{z}; 0, I_n) d\mathbf{z} \tag{1.4}$$

and the dual integral

$$I_2 = \int_{\bar{\Omega}'} h'(\mathbf{z}) f(\mathbf{z}; 0, I_n) d\mathbf{z}. \tag{1.5}$$

1.2 Orthogonal Rotation

To solve the preceding expressions for I_1 and I_2 we construct an orthonormal rotation matrix $R \in SO(n)$ such that the vector $\mathbf{b}$ is rotated to lie along the unit vector $\mathbf{e}_1 = (1, 0, ..., 0)$. Thus, we require that

$$R\mathbf{b} = |\mathbf{b}|\mathbf{e}_1,$$

where $|\mathbf{b}|$ is the norm of $\mathbf{b}$. Thus,

$$\sum_{i=1}^{n} R_{1i} b_i = \sqrt{b_1^2 + \cdot + b_n^2},$$

and hence,

$$R_{1i} = \frac{b_i}{\sqrt{b_1^2 + \cdot + b_n^2}}. \tag{1.6}$$

The remaining rows of R are arbitrary provided that R is an orthonormal rotation matrix. When an explicit construction of the remaning rows of R is needed, we construct $n - 1$ unit vectors $\mathbf{u}_2, ..., \mathbf{u}_n$ that are mutually orthonormal and orthogonal to $\mathbf{b}$. That construction can be done using Gramm-Shmidt or other algorithms. Nonetheless, when constructed, we set the jth row of R equal to $\mathbf{u}_j$.

We now let $\mathbf{w} = R\mathbf{z}$, and hence, $\mathbf{z} = R'\mathbf{w}$. Because $R'R = I_n$, it follows immediately by the linear transformation properties of the multivariate normal distribution that $f(\mathbf{z}; 0, I_n)$ is transformed into another $\mathcal{N}(0, I_n)$ distribution $f(\mathbf{w}_n; 0, I_n)$. Furthermore, the defining hyperplane of the subspace Ω' becomes

$$d - \mathbf{a} \cdot \mu - \mathbf{b} \cdot R'\mathbf{w} = 0.$$

But

$$\mathbf{b} \cdot R'\mathbf{w} = (R\mathbf{b}) \cdot \mathbf{w} = |\mathbf{b}|w_1,$$

and thus, the hyperplane equation becomes

$$d - \mathbf{a} \cdot \mu - |\mathbf{b}|w_1 = 0. \tag{1.7}$$

The transformed hyperplane now defines the new subspace

$$\Omega'' = \{\mathbf{w} : w_1 < (d - \mathbf{a} \cdot \mu)/|\mathbf{b}|\}.$$

The rotation also defines a new function

$$h''(\mathbf{w}) = h'(R'\mathbf{w}) = h(TR'\mathbf{w} + \mu).$$

Letting $U = (d - \mathbf{a} \cdot \mu)/|\mathbf{b}|$, we get

$$I_1 = \int_{\Re^{n-1}} f(\mathbf{w}_{n-1}; 0, I_{n-1}) d\mathbf{w}_{n-1} \int_{-\infty}^{U} h''(\mathbf{w}) f(w_1; 0, 1) dw_1, \tag{1.8}$$

and similarly,

$$I_2 = \int_{\Re^{n-1}} f(\mathbf{w}_{n-1}; 0, I_{n-1}) d\mathbf{w}_{n-1} \int_{U}^{\infty} h''(\mathbf{w}) f(w_1; 0, 1) dw_1, \tag{1.9}$$

where $\mathbf{w}_{n-1} = (w_2, ..., w_n)$.

2 Special Cases of h

We explicitly discuss three special cases for the function h. In the first case, h is a linear function of $\mathbf{x}$. In the second case, h is a rational function of $\mathbf{x}$ and in the third case h is an exponential function of $\mathbf{x}$ with either linear or quadratic terms.

2.1 Linear Function h

We consider the important case when the function h is a linear function of $\mathbf{x}$. Thus, we can write h as follows

$$h(\mathbf{x}) = c - \mathbf{q} \cdot \mathbf{x}$$

where $\mathbf{q}$ is a vector of n coefficients. Thus, we get that

$$h''(\mathbf{w}) = h(TR'\mathbf{w} + \mu) = c - \mathbf{q} \cdot \mu - (RT'\mathbf{q}) \cdot \mathbf{w}.$$

Let $\mathbf{s} = RT'\mathbf{q}$. Thus, Equations 1.8 and 1.9 reduce to

$$I_1 = (c - \mathbf{q} \cdot \mu)\Im[-\infty, U] - s_1 \Im_x[-\infty, U], \tag{2.1}$$

and

$$I_2 = (c - \mathbf{q} \cdot \mu)\Im[U, \infty] - s_1 \Im_x[U, \infty], \tag{2.2}$$

where

$$s_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} R_{1i} T'_{ij} q_j. \tag{2.3}$$

But by Equation 1.6, $R_{1i} = b_i/|\mathbf{b}|$, and since $\mathbf{b} = T'\mathbf{a}$ and $TT' = \Sigma$ we get that

$$s_1 = \frac{\mathbf{a}'\Sigma\mathbf{q}}{\sqrt{\mathbf{a}'\Sigma\mathbf{a}}} \tag{2.4}$$

2.2 Rational Function h

Let h denote a polynomial function of $\mathbf{x}$. As before, we can express $h''(\mathbf{w}) = h(TR'\mathbf{w} + \mu)$, where it follows trivially that h remains a polynomial function of $\mathbf{w}$ under the transformation. Thus, it suffices to show how to evaluate I_1 and I_2 for the general term $h''(\mathbf{w}) = w_1^{p_1} \cdots w_n^{p_n}$ where p_i represent arbitrary integers. We note that each ith term can be integrated separately and thus evaluation of I_1 or I_2 for $h''(\mathbf{w}) = w_1^{p_1} \cdots w_n^{p_n}$ reduces to the product of terms

$$\int w^k f(w; 0, 1) dw = -\frac{1}{\sqrt{\pi}} 2^{-\frac{3}{2} + \frac{1+k}{2}} \Gamma\left[\frac{1+k}{2}, \frac{w^2}{2}\right]$$

where Γ denotes the incomplete Gamma function.

2.3 Exponential Function h

Let h denote the exponential function $h(\mathbf{x}) = e^{c - \mathbf{q} \cdot \mathbf{x}}$. Thus, we can write

$$h''(\mathbf{w}) = e^{c - \mathbf{q} \cdot \mu} \cdot e^{-(RT'\mathbf{q}) \cdot \mathbf{w}}.$$

But for any scalar s and $N(0,1)$ distributed random variable x we get

$$\int e^{-sx} f(x;0,1)dx = e^{s^2/2} \int f(x;s,1)dw. \tag{2.5}$$

Thus, we get that

$$\int_{-\infty}^{\infty} e^{-sx} f(x;0,1)dx = e^{s^2/2} \tag{2.6}$$

and

$$\int_{-\infty}^{U} e^{-sx} f(x;0,1)dx = e^{s^2/2}\Im[-\infty, U]. \tag{2.7}$$

Thus, letting $\mathbf{s} = RT'\mathbf{q}$, and observing that $\mathbf{s}\cdot\mathbf{s} = \mathbf{q}'TR'RT'\mathbf{q} = \mathbf{q}'\Sigma\mathbf{q}$ we get that

$$I_1 = e^{c-\mathbf{q}\,\mu+\frac{1}{2}\mathbf{q}'\Sigma\mathbf{q}}\Im[-\infty, U] \tag{2.8}$$

and

$$I_2 = e^{c-\mathbf{q}\,\mu+\frac{1}{2}\mathbf{q}'\Sigma\mathbf{q}}\Im[U, \infty]. \tag{2.9}$$

The final case when the function h is of the form

$$h(\mathbf{x}) = e^{\mathbf{x}'K\mathbf{x}+\mathbf{q}'\mathbf{x}+c}$$

can also be solved, and is left as an excercise to the interested reader.

3 The Induced Multivariate Normal Distribution

As before, let $f(\mathbf{x};\boldsymbol{\mu},\boldsymbol{\Sigma})$ denote the multivariate normal density function of $\mathbf{x} \in \Re^n$ with mean vector μ and covariance matrix Σ. Let $\mathbf{y}$ denote an m dimensional vector related to $\mathbf{x}$ by the m constraint equations

$$y_i - \mathbf{a}_i \cdot \mathbf{x} = 0, \tag{3.1}$$

where $i = 1, ..., m$. In this section we derive the distribution of $\mathbf{y}$ induced by the distribution on $\mathbf{x}$ and the m constraint equations.

Let $p \leq n$ denote the dimension of the vector space spanned by the m vectors $\mathbf{a}_1, ..., \mathbf{a}_m$. We assume that the first p vectors $\mathbf{a}_1, ..., \mathbf{a}_p$ are linearly independent, and therefore, that they span that space. Otherwise, we can reorder the vectors appropriately. In deriving the distribution of

y we only need to derive a distribution of $\mathbf{y}_p = (y_1, ..., y_p)$ related to $\mathbf{x}$ by the first p constraint equations in Equation 3.1. This result follows because for $k > p$, each y_k can be expressed as a linear combination of the y_i, $i \leq p$.

Letting A denote the $p \times n$ matrix where the ith row of A is the vector $\mathbf{a}_i$, we can rewrite the p constraint equations as

$$\mathbf{y}_p - A\mathbf{x} = 0. \tag{3.2}$$

Let $g(\mathbf{y}_p)$ denote the induced distribution of $\mathbf{y}_p$. For a given $\mathbf{y}_p$, let $\Omega(\mathbf{y}_p)$ denote the subspace of $\mathbf{x} \in \Re^n$ that satisfies the p constraint equations for $\mathbf{y}_p$. Thus, it follows that

$$g(\mathbf{y}_p) = \int_{\Omega(\mathbf{y}_p)} f(\mathbf{x}; \mu, \Sigma) d\mathbf{x}. \tag{3.3}$$

To obtain a solution for the induced distribution g we must solve the integral in Equation 3.3. Observe that the space $\Omega(\mathbf{y}_p)$ is the subspace generated by the intersection of the p hyperplanes defined in Equation 3.2. We procede as follows. As before, we perform a linear transformation of the space using the Cholesky decomposition of Σ, and thus transforming the distribution $f(\mathbf{x}; \mu, \Sigma)$ into an $\mathcal{N}(0, I_n)$ distribution. Because the transformation is linear, the space $\Omega(\mathbf{y}_p)$ is transformed into a new space that is the intersection of the transformed hyperplanes defined by the p transformed constraint equations. We next prove that the new space $\Omega'(\mathbf{y}_p)$ can be generated by the intersection of a set of p orthogonal hyperplanes. We can now rotate these orthogonal hyperplanes so that they lie orthogonal to coordinate axes, while leaving the $\mathcal{N}(0, I_n)$ distribution invariant. Once rotated, we can complete the integration.

As before, we let $\mathbf{z} = T^{-1}(\mathbf{x} - \mu)$, where $TT' = \Sigma$ denotes the Cholesky decomposition of Σ. Thus, we get p new constraint equations

$$y_i - \mathbf{a}_i \cdot \mu - \mathbf{b}_i \cdot \mathbf{z} = 0, \tag{3.4}$$

where $\mathbf{b}_i = T'\mathbf{a}_i$ and $i = 1, ..., p$. The subspace $\Omega(\mathbf{y}_p)$ is transformed into the space $\Omega'(\mathbf{y}_p)$ defined as the subspace of $\mathbf{z}$ that satisfies the constraint equations

$$\mathbf{y}_p - A\mu - AT\mathbf{z} = 0.$$

Furthermore, the vector $\mathbf{z}$ has distribution density function $f(\mathbf{z}; 0, I_n)$, and thus

$$g(\mathbf{y}_p) = \int_{\Omega'(\mathbf{y}_p)} f(\mathbf{z}; 0, I_n) d\mathbf{z}. \tag{3.5}$$

Let the vectors $\mathbf{u}_1, ..., \mathbf{u}_p$ form an orthonormal basis that spans the space of $\mathbf{b}_1, ..., \mathbf{b}_p$. Thus, for

$i = 1, ..., p$.

$$\mathbf{b}_i = \sum_{j=1}^{p} \beta_{ij} \mathbf{u}_j, \tag{3.6}$$

where $\beta_{ij} = \mathbf{b}_i \cdot \mathbf{u}_j$. Let Δ denote the $p \times p$ matrix of β_{ij}. Let U denote the $p \times n$ matrix with ith row containing $\mathbf{u}_i$ for all $i = 1, ..., p$. Thus, we can rewrite Equation 3.4 as

$$\mathbf{y}_p - A\mu - \Delta U \mathbf{z} = 0, \tag{3.7}$$

or, since Δ is invertible, as

$$\Delta^{-1}\mathbf{y}_p - \Delta^{-1}A\mu - U\mathbf{z} = 0. \tag{3.8}$$

Now let $\mathbf{w} = R\mathbf{z}$ where R is an orthonormal rotation such that $R\mathbf{u}_i = \mathbf{e}_i$ for $i = 1, ..., p$ where $\mathbf{e}_i$ denotes the ith unit basis vector. Observe that UR' is the $p \times n$ matrix with ijth element equal to δ_{ij} for $j \le i$, and zero otherwise. Thus, $UR'\mathbf{w} = \mathbf{w}_p$ where $\mathbf{w}_p = (w_1, ..., w_p)$. Substituting $\mathbf{z} = R'\mathbf{w}$ in Equation 3.8 we get

$$\mathbf{w}_p = \Delta^{-1}\mathbf{y}_p - \Delta^{-1}A\mu. \tag{3.9}$$

Thus, the subspace $\Omega'(\mathbf{y}_p)$ is transformed into the subspace

$$\Omega''(\mathbf{y}_p) = \{\mathbf{w} : \mathbf{w}_p = \Delta^{-1}\mathbf{y}_p - \Delta^{-1}A\mu\},$$

and the density function $f(\mathbf{z}; 0, I_n)$ is transformed into the density function $f(\mathbf{w}; 0, I_n)$. Thus, we get that

$$\begin{aligned} g(\mathbf{y}_p) &= \int_{\Omega''(\mathbf{y}_p)} f(\mathbf{w}; 0, I_n) d\mathbf{w} \\ &= f(\Delta^{-1}\mathbf{y}_p - \Delta^{-1}A\mu; 0, I_p) \\ &= f(\mathbf{y}_p; \Delta\Delta', A\mu), \end{aligned}$$

where the last equality follows from well known properties of the multivariate normal density function. Thus, the density function for $\mathbf{y}_p$ induced by the multivariate normal distribution on $\mathbf{x}$ and the p independent constraint equations is *another* multivariate normal distribution with covariance matrix $\Delta\Delta'$ and mean vector $A\mu$.

4 Expectation of General Functions Over Polyhedral Subspaces

We revisit the integral I_1 in Equation 1.1. We are now interested in the solution of I_1 when the subspace of integration Ω is a polyhedra described as the region enclosed by the intersection of multiple halfspaces,

$$d_i - \mathbf{a}_i \cdot \mathbf{x} > 0, \tag{4.1}$$

for $i = 1, ..., m$, or equivalently,

$$\mathbf{d} - A\mathbf{x} > 0, \tag{4.2}$$

where A is an $m \times n$ matrix with ith row equal to $\mathbf{a}_i$. Let $\Omega_i = \{\mathbf{x} : d_i - \mathbf{a}_i \cdot \mathbf{x} > 0\}$, and therefore, $\Omega = \cap_{i=1}^{m} \Omega_i$. Observe that we allow the d_i and $\mathbf{a}_i$ to be negative, that is,

$$-|d_i| + |\mathbf{a}_i| \cdot \mathbf{x} > 0,$$

and hence, we may include the complement halfspaces $\hat{\Omega}_i = \{\mathbf{x} : |d_i| - |\mathbf{a}_i| \cdot \mathbf{x} < 0\}$ in the definition of the polyhedra Ω. (Observe that in the preceding expressions, $|\mathbf{a}_i|$ represents the vector $\mathbf{a}_i$ with all positive entries, and not its norm.)

Proceeding similarly to the derivation in Section 1, we perform a Cholesky decomposition $TT' = \Sigma$, enabling us to reduce I_1 to

$$I_1 = \int_{\Omega'} h'(\mathbf{z}) f(\mathbf{z}; 0, I_n) d\mathbf{z} \tag{4.3}$$

where $h'(\mathbf{z}) = h(T\mathbf{z} + \mu)$ and $\Omega' = \cap_{i=1}^{m} \Omega'_i$ where

$$\Omega'_i = \{\mathbf{z} : d_i - \mathbf{a}_i \cdot \mu - \mathbf{b}_i \cdot \mathbf{z} > 0\},$$

and $\mathbf{b}_i = T'\mathbf{a}_i$.

Equation 4.3 cannot be solved exactly in closed form. In fact, we show in Section 6 that the solution of I_1 is NP-hard. We can approximate I_1 as follows. We identify a set of mutually orthogonal halfspaces that best approximates the polyhedra Ω'. Let

$$\Omega''_i = \{\mathbf{z} : c_i - \mathbf{u}_i \cdot \mathbf{z} > 0\},$$

for $i = 1, ..., r$, denote those r orthogonal halfspaces, and let $\Omega'' = \cap_{i=1}^{r} \Omega''_i$ denote the polyhedra defined by the orthogonal halfspaces. By construction, the vectors $\mathbf{u}_i$ are orthogonal. Furthermore, we assume that they have been normalized, and hence are orthonormal vectors. We can express Ω''

as the solution of

$$\mathbf{c} - U\mathbf{z} > 0, \tag{4.4}$$

where U is the $r \times n$ matrix with ith row equal to $\mathbf{u}_i$. Observe that $r \le p$, where p is the dimension of the subspace spanned by the vectors $\mathbf{b}_i$ (or equivalently, by the vectors $\mathbf{a}_i$). With the preceding approximation of the polyhedra Ω', we have that

$$I_1' = \int_{\Omega''} h'(\mathbf{z}) f(\mathbf{z}; 0, I_n) d\mathbf{z} \tag{4.5}$$

approximates I_1. We now show how to solve I_1' exactly. In the next section we show how to construct the halfspaces Ω_i''.

We construct a rotation matrix $R \in SO(n)$ such that $R\mathbf{u}_i = \mathbf{e}_i$ for $i = 1, ..., r$ where $\mathbf{e}_i$ is the ith unit vector. For $i = 1, ..., r$, the ith row of R is equal to $\mathbf{u}_i$. For $i > r$, the rows of R are arbitrary subject to the constraint that R is a rotation matrix. When explicit construction of those rows is necessary, we construct $n - r$ orthonormal vectors $\mathbf{v}_i$ that are orthogonal to the vectors $\mathbf{u}_i$. That construction can be done using the Gram-Schmidt algorithm. The ith row of R for $i > r$ is set equal to the vector $\mathbf{v}_{i-r}$ completing the construction of R.

Let $\mathbf{w} = R'\mathbf{z}$. This rotation transforms the constraints in Equation 4.4 into

$$\mathbf{c} - UR'\mathbf{w} > 0. \tag{4.6}$$

But by construction of R, UR' is the $r \times n$ matrix with ijth element equal to δ_{ij} for $j \le i$ and zero otherwise. Thus, the halfspaces Ω_i'' are transformed into $\hat{\Omega}_i = \{\mathbf{w} : w_i < c_i\}$. Furthermore, the density function $f(\mathbf{z}; 0, I_n)$ is transformed into $f(\mathbf{w}; 0, I_n)$ and the function h' becomes $h''(\mathbf{w}) = h'(R'\mathbf{w}) = h(TR'\mathbf{w} + \mu)$. Thus, I_1' becomes

$$I_1' = \int_{\Re^{n-r}} \prod_{i=1}^{r} \int_{-\infty}^{c_i} h''(\mathbf{w}) f(\mathbf{w}; 0, I_n) d\mathbf{w} \tag{4.7}$$

4.1 Construction of Halfspaces Ω_i''

Recall that we desire to construct a set of orthogonal halfspaces Ω_i'', $i = 1, ..., r$, that best approximate the polyhedra Ω' enclosed by the intersection of the m halfspaces

$$\Omega_i' = \{\mathbf{z} : d_i - \mathbf{a}_i \cdot \mu - \mathbf{b}_i \cdot \mathbf{z} > 0\}$$

Let $\mathbf{v}_i$, $i = 1, ..., p$ denote an orthonormal vector basis that spans the p-dimensional subspace defined by the vectors $\mathbf{b}_i$, $i = 1, ..., m$. Let $\hat{\mathbf{b}}_i$ denote the normalized vectors $\mathbf{b}_i$. We can express each

$\hat{\mathbf{b}}_i$ as

$$\hat{\mathbf{b}}_i = \sum_{j=1}^{p} \lambda_{ij} \mathbf{v}_j ,$$

where $\lambda_{ij} = \hat{\mathbf{b}}_i \cdot \mathbf{v}_j$. To obtain a good approximation of the polyhedra defined by the Ω_i' using the orthogonal halfspaces Ω_i'' we would like the coefficients λ_{ij} to be either large (close to one) or small (close to zero). For example, if for each i, λ_{ij} equals one for some j and zero otherwise, then the orthogonal halfspaces Ω_i'' exactly reproduce the polyhedra Ω'. (This result trivally follows by noting that the $\hat{\mathbf{b}}_i$ already form an orthonormal set of vectors and hence the halfspaces Ω_i' are orthogonal.) To achieve an optimal set of coefficients λ_{ij} we can rotate the vectors $\mathbf{v}_j$ so that they are approximately oriented along the $\hat{\mathbf{b}}_i$.

Let Λ denote the $m \times p$ matrix with ijth element equal to λ_{ij}. We rotate the vectors $\mathbf{v}_j$ by a matrix R such that the new coefficients $\Lambda^* = \Lambda R$ are either small or large. The rotation can be found by minimizing the well known criterion

$$V = \sum_{j=1}^{p} \sum_{i=1}^{m} (\lambda_{ij}^*)^4 - \frac{\gamma}{m} \sum_{j=1}^{p} \left[\sum_{i=1}^{m} (\lambda_{ij}^*)^2 \right]^2 . \tag{4.8}$$

Equation 4.8 is well known in factor analysis, and minimization of V when $\gamma = 1$ yields the *varimax* rotation and when $\gamma = 0$ yields *qartimax* rotation.

We now define the new orthonormal basis $\mathbf{u}_i = R'\mathbf{v}_i$, $i = 1, ..., p$, and thus

$$\hat{\mathbf{b}}_i = \sum_{j=1}^{p} \lambda_{ij}^* \mathbf{u}_j . \tag{4.9}$$

We approximate each $\hat{\mathbf{b}}_i$ by the $\mathbf{u}_j$ with largest coefficient λ_{ij}^*. In general, if several $\hat{\mathbf{b}}_i$ are approximately collinear, then we do not need the entire basis $\mathbf{u}_j$, $j = 1, ..., p$ to approximate the vectors $\hat{\mathbf{b}}_i$. Assume that $r \leq p$ basis vectors are required. We further assume that the vectors $\mathbf{u}_j$ have been reordered so that $\mathbf{u}_1, ..., \mathbf{u}_r$ define the desired r basis vectors. We define the mapping σ from the index set $\{1, ..., m\}$ into the index set $\{1, ..., r\}$ such that $\mathbf{u}_{\sigma(i)}$ is the basis vector that best approximates $\hat{\mathbf{b}}_i$. We can now define the m orthogonal halfspaces Ω_i'' as

$$\Omega_i'' = \{\mathbf{z} : d_i - \mathbf{a}_i \cdot \mu - |\mathbf{b}_i| \mathbf{u}_{\sigma(i)} \cdot \mathbf{z} > 0\} \tag{4.10}$$

Note that the goodness of this approximation rest largely on how well the varimax or quartimax rotations can generate a matrix Λ^* with entries that are either close to one or close to zero.

Because $r < m$, all vectors $\hat{\mathbf{b}}_i$ such that $i \in \sigma^{-1}(j)$ will get mapped into the same basis vector $\mathbf{u}_j$. Thus, those halfspaces Ω_i'' will have parallel boundaries, and it follows trivially that their

intersection is the halfspace Ω''_k where $k \in \sigma^{-1}(j)$ is the index given by

$$\mathrm{argmin}_{i\in\sigma^{-1}(j)} \left[\frac{d_i - \mathbf{a}_i \cdot \mu}{|\mathbf{b}_i|}\right]. \tag{4.11}$$

Thus, for Equation 4.4, we get

$$c_j = \min_{i\in\sigma^{-1}(j)} \left[\frac{d_i - \mathbf{a}_i \cdot \mu}{|\mathbf{b}_i|}\right] \tag{4.12}$$

and U is the $r \times n$ matrix with ith row equal to $\mathbf{u}_i$.

5 Special Cases of h for Polyhedral Integration

We explicitly discuss the cases when h is a linear function of $\mathbf{x}$ and when it is an exponential function of $\mathbf{x}$.

5.1 Linear Function h

As before, we express h as

$$h(\mathbf{x}) = f - \mathbf{q} \cdot \mathbf{x}$$

where $\mathbf{q}$ is a vector of n coefficients.

$$h''(\mathbf{w}) = h(TR'\mathbf{w} + \mu) = f - \mathbf{q} \cdot \mu - (RT'\mathbf{q}) \cdot \mathbf{w}.$$

Let $\mathbf{s} = RT'\mathbf{q}$. Thus, Equation 4.7 reduces to

$$I_1' = (f - \mathbf{q} \cdot \mu) \prod_{i=1}^{r} \Im[-\infty, c_i] - \sum_{j=1}^{r} s_j \Im_x[-\infty, c_j] \prod_{i=1, i\neq j}^{r} \Im[-\infty, c_i] \tag{5.1}$$

where for $j \leq r$

$$s_j = \sum_{i=1}^{n}\sum_{k=1}^{n} R_{ji} T'_{ik} q_k = \mathbf{u}'_j T' \mathbf{q}, \tag{5.2}$$

since by construction, the jth row of R is $\mathbf{u}_j$ for $j \leq r$. Thus, if we let $\mathbf{s}_p$ denote the vector containing the first p components of $\mathbf{s}$, we get

$$\mathbf{s}_p = UT'\mathbf{q}.$$

In the special case when the vectors $\mathbf{b}_j$, $j = 1, ..., m$ (and hence $r = m$) are mutually orthogonal, we get

$$s_j = \frac{\mathbf{a}_j' \Sigma \mathbf{q}}{\sqrt{\mathbf{a}_j' \Sigma \mathbf{a}_j}}, \tag{5.3}$$

since $\mathbf{b}_j = T'\mathbf{a}_j$ and $TT' = \Sigma$.

5.2 Exponential Function h

Let h denote the exponential function $h(\mathbf{x}) = e^{f - \mathbf{q} \cdot \mathbf{x}}$. Thus, we can write

$$h''(\mathbf{w}) = e^{f - \mathbf{q} \cdot \boldsymbol{\mu}} \cdot e^{-(RT'\mathbf{q}) \cdot \mathbf{w}}.$$

Thus, letting $\mathbf{s} = RT'\mathbf{q}$, and using Equations 2.6 and 2.7 we get

$$I_1' = e^{f - \mathbf{q} \cdot \boldsymbol{\mu} + \frac{1}{2}\mathbf{q}'\Sigma\mathbf{q}} \prod_{i=1}^{r} \Im[-\infty, c_i] \tag{5.4}$$

where we used the relationship $\mathbf{s} \cdot \mathbf{s} = \mathbf{q}'TR'RT'\mathbf{q} = \mathbf{q}'\Sigma\mathbf{q}$.

6 Integration of the Multivariate Normal Distribution over Polyhedral Subspaces is NP-hard

Much research has been dedicated to solve, or efficiently approximate, the integral $f(\mathbf{x}; \boldsymbol{\mu}, \Sigma)$ over an arbitray region. Even the simple bounding region $x_i < 0$ for $i = 1, ..., n$, has eluded mathematicians. We prove that the problem of integrating even $f(\mathbf{x}; 0, I_n)$ over an arbitrary polyhedra is NP-hard. The proof consists of a straightforward reduction of the NP-hard problem of computing the volume of a polyhedra. Let $\mathbf{a}_i \cdot \mathbf{x} < 0$, $i = 1, ..., p$ define the polyhedra Ω. The volume enclosed by that polyhedra, $\mathrm{vol}(\Omega)$ can be easily shown to equal

$$\mathrm{vol}(\Omega) = \int_{\Omega} f(\mathbf{x}; 0, I_n) d\mathbf{x},$$

thus proving our claim.

7 Appendix 1

We present some well known results on the integration of a normal distribution. Let $f(x;\mu,\sigma^2)$ denote the normal distribution with mean μ and variance σ^2. Let $q \equiv q(x) = (x-\mu)/\sigma$. Then,

$$\int f(x;\mu,\sigma^2)dx = \frac{1}{2}\mathrm{Erf}\left[\frac{q}{\sqrt{2}}\right], \tag{7.1}$$

and

$$\int xf(x;\mu,\sigma^2)dx = -\frac{\sigma}{\sqrt{2\pi}}e^{-q^2/2} + \frac{\mu}{2}\mathrm{Erf}\left[\frac{q}{\sqrt{2}}\right]. \tag{7.2}$$

We let $\Im[u,v]$ and $\Im_x[u,v]$ denote, respectively, the integrals in Equations 7.2 and 7.1 evaluated between the limits $u \leq x \leq v$. Also observe that these integrals trivially satisfy the following properties

$$\begin{aligned}
&\Im[u,v] = \Im[-\infty,v] - \Im[-\infty,u] \\
&\Im[-\infty,u] + \Im[u,\infty] = 1 \\
&\Im_x[u,v] = \Im_x[-\infty,v] - \Im_x[-\infty,u] \\
&\Im_x[-\infty,u] + \Im[u,\infty] = \mu
\end{aligned}$$

We now evaluate the derivatives of the integrals $\Im$ and $\Im_x$. From the above properties, it suffices to evaluate the integrals $\Im[-\infty,v]$ and $\Im_x[-\infty,v]$. Letting $q \equiv q(v) = \frac{v-\mu}{\sigma}$,

$$\frac{d}{dv}\Im[-\infty,v] = \frac{1}{\sigma\sqrt{2\pi}}e^{-q^2/2} = f(v;\mu,\sigma^2), \tag{7.3}$$

and

$$\frac{d}{dv}\Im_x[-\infty,v] = \left(\frac{\mu}{\sigma\sqrt{2\pi}} + \frac{q}{\sqrt{2\pi}}\right)e^{-q^2/2} = v\cdot f(v;\mu,\sigma^2). \tag{7.4}$$

References

[1] Y.L. Tong, **The Multivariate Normal Distribution**, 1990, Springer-Verlag, New York.

1 Introduction

This report focuses on several incarnations of Rapt Technologies proprietary *uniform production policy* for production planning and component allocation under scarce resources. In this report we revisit the definition of the policy for the MCMP model (previously defined in [1, 2]) and we derive closed form analytical solutions to the expectation of a multivariate normal distribution under this policy. What we are seeking in this paper is an expression for the expected production vector given a demand distribution (which in general, will not be fully satisfied in all situations).

2 Background

Let $f(\mathbf{x}; \boldsymbol{\mu}, \Sigma)$ denote the multivariate normal density function of $\mathbf{x} \in \Re^n$ with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ. Let $[m] = \{1, ..., m\}$. Let $h(\mathbf{x})$ denote an arbitrary multivariate functions of $\mathbf{x}$ and let Ω denote the polyhedra described as the region enclosed by the intersection of multiple half spaces,

$$d_i - \mathbf{a}_i \cdot \mathbf{x} > 0, \tag{2.1}$$

for $i \in [m]$. Equivalently, define

$$\mathbf{d} - A\mathbf{x} > 0, \tag{2.2}$$

where A is an $m \times n$ matrix with ith row equal to $\mathbf{a}_i$. Let $\Omega_i = \{\mathbf{x} : d_i - \mathbf{a}_i \cdot \mathbf{x} > 0\}$, and therefore, $\Omega = \cap_{i \in [m]} \Omega_i$.

The space Ω denotes the *feasible* region where the component level $\mathbf{d}$ is sufficient to meet any demand $\mathbf{x}$ in that space. When $\mathbf{x}$ falls outside of that space, then we must invoke either a *production policy* that prescribes the level of products to manufacture while satisfying the component levels, or equivalently, an *allocation policy* that prescribes how to redistribute the components among the products. The *uniform policy* represents a production policy, rather than an allocation policy. In general, production policies are preferred over allocation policies since company revenue is linked directly to product manufacture and not component consumption.

A *production policy* $\mathbf{y}$ is a mapping from the space $\bar{\Omega}$ into the space Ω such that for any $\mathbf{x} \in \bar{\Omega}$ we have that $\mathbf{y}(\mathbf{x}) \leq \mathbf{x}$. A *maximal production policy* is a production policy that maps $\bar{\Omega}$ into the boundary subspace of Ω. Thus, a maximal production policy will deplete at least one component.

In TR-02-99 we obtain closed-from expressions for the following general integral

$$I(\mathbf{d}) = \int_{\Omega} \mathbf{x} f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) d\mathbf{x}, \tag{2.3}$$

and

$$\bar{I}(\mathbf{d}) = \int_{\bar{\Omega}} \mathbf{x} f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) d\mathbf{x} \tag{2.4}$$

by noting that

$$\bar{I}(\mathbf{d}) = \mathbf{E}\mathbf{x} - I(\mathbf{d}) = \boldsymbol{\mu} - I(\mathbf{d}). \tag{2.5}$$

TR-02-99 also solved the above integrals for the priority allocation scheme (or equivalently, the priority production policy). In this report we extend the development to include other production policies.

2.1 Using these results

While this paper only presents the calculation of the expected production vector, this result can be directly used to evaluate the expected value of any linear function of the production.

3 The u-policy

Let $\mathbf{u}$ denote some arbitrary normalized vector. The *(u-policy)* $\mathbf{y}$ is defined for all as

$$\mathbf{y}(\mathbf{x}) = \begin{cases} \mathbf{x} - \alpha(\mathbf{x})\mathbf{u} & \text{if } \mathbf{x} \in \bar{\Omega} \\ \mathbf{x} & \text{otherwise} \end{cases} \tag{3.1}$$

for some scalar valued function α such that

$$\mathbf{d} - A\mathbf{q} > 0, \tag{3.2}$$

and

$$\mathbf{q}(\mathbf{x}) \leq \mathbf{x}. \tag{3.3}$$

The choice of vector $\mathbf{u}$ and function α are arbitrary, but in general, they can be chosen to optimize some objective function.

The most important characteristic of the u-policy is that it prescribes a particular direction for the $\mathbf{x} - \mathbf{q}(\mathbf{x})$ vector, irrespective of the value of $\mathbf{x}$. This can lead to unexpected prescriptions, such as a negative production corresponding to a positive demand vector.

3.1 Transformation of the Integral I

In TR-02-99 we show that after several transformations, the general integral $I(h \circ \mathbf{q}, \mathbf{d})$ can be reduced to the following equivalent integral

$$I(h \circ \mathbf{q}, \mathbf{d}) = \int_{\Re^{n-r}} \prod_{i=1}^{r} \int_{-\infty}^{c_i} h''(\mathbf{w}) f(\mathbf{w}; 0, I_n) d\mathbf{w} \tag{3.4}$$

where $h''(\mathbf{w}) = h(\mathbf{q}(TR'\mathbf{w} + \boldsymbol{\mu}))$,

$$c_i = \min_{j \in \sigma^{-1}(i)} \left[\frac{d_j - \mathbf{a}_j \cdot \boldsymbol{\mu}}{\sqrt{\mathbf{a}_j' \Sigma \mathbf{a}_j}} \right], \tag{3.5}$$

and σ is the mapping from the index set $[m]$ into the set $[r]$ defined in TR-02-99.

3.2 Determining of α

Before we show how to compute the integral

$$\int_{\bar{\Omega}} \alpha(\mathbf{x}) f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) d\mathbf{x} = \mathbf{E}\alpha - \int_{\Omega} \alpha(\mathbf{x}) f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) d\mathbf{x}, \tag{3.6}$$

we need to define α. For linear, polynomial, or exponential forms of α this computation can be done using the methods derived in TR-02-99. For the uniform production policy however we choose

$$\alpha(\mathbf{x}) = \max_{i \in [m]} \left[\frac{\mathbf{a}_i \cdot \mathbf{x} - d_i}{\mathbf{a}_i \cdot \mathbf{u}} \right]. \tag{3.7}$$

This choice guarantees that for any $\mathbf{x} \in \bar{\Omega}$ and a given $\mathbf{u}$, the production level $\mathbf{q}(\mathbf{x})$ is the highest level compatible with the restrictions in Equations 3.2 and 3.3.

Using the transformations of TR-02-99 we can reduce $\alpha(\mathbf{x})$ as follows. First, letting $\mathbf{x} = T\mathbf{z} + \boldsymbol{\mu}$ we get

$$\alpha'(\mathbf{z}) = \alpha(T\mathbf{z} + \boldsymbol{\mu}) = \max_{i \in [m]} \left[\frac{\mathbf{b}_i \cdot \mathbf{z} + \mathbf{a}_i \cdot \boldsymbol{\mu} - d_i}{\mathbf{a}_i \cdot \mathbf{u}}, \right] \tag{3.8}$$

where $\mathbf{b}_i = T'\mathbf{a}_i$. Next, from the orthomax rotation used to 'orthonormalize' the space Ω we have that the m vectors $\mathbf{b}_i$ can be approximated by the r orthonormal vectors $\mathbf{u}_j$ such that $\mathbf{b}_i \simeq |\mathbf{b}_i| \mathbf{u}_{\sigma(i)}$. As before, σ is a mapping from the index set $[m]$ into $[r]$. Substituting for the $\mathbf{b}_i$ and defining the rotation matrix R such that $R\mathbf{u}_{\sigma(i)} = \mathbf{e}_{\sigma(i)}$, where $\mathbf{e}_i$ are unit vectors, we get

$$\alpha''(\mathbf{w}) = \alpha(R'\mathbf{w}) = \max_{i \in [m]} \left[\beta_i w_{\sigma(i)} + \gamma_i \right] \tag{3.9}$$

where, noting that $|\mathbf{b}_i| = \sqrt{\mathbf{a}_i' \Sigma \mathbf{a}_i}$,

$$\beta_i = \frac{\sqrt{\mathbf{a}_i' \Sigma \mathbf{a}_i}}{\mathbf{a}_i \cdot \mathbf{u}}$$

and

$$\gamma_i = \frac{\mathbf{a}_i \cdot \boldsymbol{\mu} - d_i}{\mathbf{a}_i \cdot \mathbf{u}}.$$

We can rewrite α'' as

$$\alpha''(\mathbf{w}) = \max_{i \in [r]} \left[\max_{j \in \sigma^{-1}(i)} (\beta_j w_i + \gamma_j) \right]. \tag{3.10}$$

3.3 Computing I

We can now show how to compute

$$I = \int_{\Omega} \alpha(\mathbf{x}) f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) d\mathbf{x}. \tag{3.11}$$

We use the results of TR-02-99 and of the preceding section to reduce I to

$$I = \prod_{i=1}^{r} \int_{-\infty}^{c_i} \alpha''(\mathbf{w}) f(\mathbf{w}; 0, I_n) d\mathbf{w}, \tag{3.12}$$

where

$$c_i = \min_{j \in \sigma^{-1}(i)} \left[\frac{d_j - \mathbf{a}_j \cdot \boldsymbol{\mu}}{|\mathbf{b}_j|} \right]$$

as defined in TR-02-99. It now follows directly from the results in the Appendix that

$$I = \sum_{i=1}^{r} \int_{-\infty}^{c_i} \max_{j \in \sigma^{-1}(i)} (\beta_j w_i + \gamma_j) \prod_{k \neq i} \Im[-\infty, \min(c_k, U_{ik}(x))] f(x; 0, 1) dx \tag{3.13}$$

where, letting

$$M_i = \max_{j \in \sigma^{-1}(i)} (\beta_j w_i + \gamma_j),$$

we have that

$$U_{ik} = \min_{j \in \sigma^{-1}(k)} \left[\frac{M_i - \gamma_j}{\beta_j} \right].$$

The integral I is a sum of $r < m$ univariate integrals. The integral can be readily solved using numerical integration techniques such as quadratures or even Simpson approximations. Note that the integrand is a product of functions $\Im[]$ that results in a well behaved monotonic increasing function.

4 The $\hat{\mathbf{x}}$-policy

The $\hat{\mathbf{x}}$-policy is similar conceptually to the u-policy, except that demand is scaled proportionally:

$$\mathbf{q}(\mathbf{x}) = \begin{cases} \alpha(\mathbf{x})\mathbf{x} & \text{if } \mathbf{x} \in \bar{\Omega} \\ \mathbf{x} & \text{otherwise} \end{cases} \tag{4.1}$$

The $\hat{\mathbf{x}}$-policy is superior to the $\hat{\mathbf{u}}$-policy that it ensures that positive demand maps to positive production.

4.1 Determining of α for the $\hat{\mathbf{x}}$-policy

Alpha is chosen similarly to the $\hat{\mathbf{u}}$-policy case:

$$\alpha(\mathbf{x}) = \min_{i \in [m]} \left[\frac{d_i}{\mathbf{a}_i \cdot \mathbf{x}} \right]. \tag{4.2}$$

As before, first, we let $\mathbf{x} = T\mathbf{z} + \boldsymbol{\mu}$:

$$\alpha'(\mathbf{z}) = \alpha(T\mathbf{z} + \boldsymbol{\mu}) = \min_{i \in [m]} \left[\frac{d_i}{\mathbf{b}_i \cdot \mathbf{z} + \mathbf{a}_i \cdot \boldsymbol{\mu}} \right] \tag{4.3}$$

where $\mathbf{b}_i = T'\mathbf{a}_i$. Next, from the orthomax rotation used to 'orthonormalize' the space Ω we have that the m vectors $\mathbf{b}_i$ can be approximated by the r orthonormal vectors $\mathbf{u}_j$ such that $\mathbf{b}_i \simeq |\mathbf{b}_i|\mathbf{u}_{\sigma(i)}$. As before, σ is a mapping from the index set $[m]$ into $[r]$. Substituting for the $\mathbf{b}_i$ and defining the rotation matrix R such that $R\mathbf{u}_{\sigma(i)} = \mathbf{e}_{\sigma(i)}$, where $\mathbf{e}_i$ are unit vectors, we get

$$\alpha''(\mathbf{w}) = \alpha'(R'\mathbf{w}) = \min_{i \in [m]} \left[\frac{1}{\beta_i w_{\sigma(i)} + \gamma_i} \right] \tag{4.4}$$

where, noting that $|\mathbf{b}_i| = \sqrt{\mathbf{a}_i' \Sigma \mathbf{a}_i}$,

$$\beta_i = \frac{\sqrt{\mathbf{a}_i' \Sigma \mathbf{a}_i}}{d_i} \tag{4.5}$$

and

$$\gamma_i = \frac{\mathbf{a}_i \cdot \boldsymbol{\mu}}{d_i}. \tag{4.6}$$

We can rewrite α'' as

$$\alpha''(\mathbf{w}) = \min_{i \in [r]} \left[\min_{j \in \sigma^{-1}(i)} \left(\frac{1}{\beta_j w_i + \gamma_j} \right) \right]. \tag{4.7}$$

4.2 Evaluating the expected production vector for the $\hat{\mathbf{x}}$-policy

In the $\hat{\mathbf{x}}$-policy case, the integrand that is of concern is $\alpha(\mathbf{x})\mathbf{x}$. A for the u-policy, we attempt to reduce the multidimensional integral to a sum of one-dimensional ones. However, we need to make the following approximation:

$$\alpha''(\mathbf{w}) = \min_{i \in [m]} \left[\frac{1}{\beta_i w_{\sigma(i)} + \gamma_i} \right] \approx \frac{1}{\max_{i \in [m]} \left[\beta_i w_{\sigma(i)} + \gamma_i \right]}. \tag{4.8}$$

The approximation here is exact when $\beta_i w_{\sigma(i)} + \gamma_i > 0$ for all $i \in [m]$. However, it is potentially very far off otherwise. We next show that the approximation holds in the region of importance for the integral I.

Assume that the region of appreciable demand probability falls completely in the quadrant $Q = \{\mathbf{x} | x_i > 0 \forall i\}$, and that all products require a nonnegative amount of each component ($a_{ij} \geq 0 \forall i, j$). In this case, we have that

$$\beta_i w_{\sigma(i)} + \gamma_i = \mathbf{a}_i \cdot \mathbf{x} \geq 0 \forall i, \tag{4.9}$$

and hence the approximation in Equation 4.8 holds for the purposes of evaluating the integral I.

With the above transformation, the result in the Appendix applies, and the integral can be evaluated as follows:

$$\mathbf{I} = \int_{\Omega} \alpha(\mathbf{x}) \mathbf{x} f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) d\mathbf{x}. \tag{4.10}$$

We use the results of TR-02-99 and of the preceding section to reduce I to

$$\begin{aligned}
\mathbf{I} &= \int_{-\infty}^{c_1 \cdots c_r} d\mathbf{w} f(\mathbf{w}; 0, I_n)\alpha''(\mathbf{w})(TR'\mathbf{w} + \boldsymbol{\mu}) && (4.11) \\
&= TR'\mathbf{I}_1 + I_2\boldsymbol{\mu}, && (4.12) \\
\mathbf{I}_1 &= TR' \int_{-\infty}^{c_1 \cdots c_r} d\mathbf{w} f(\mathbf{w}; 0, I_n)\alpha''(\mathbf{w})\mathbf{w}, && (4.13) \\
I_2 &= \int_{-\infty}^{c_1 \cdots c_r} d\mathbf{w} f(\mathbf{w}; 0, I_n)\alpha''(\mathbf{w}) && (4.14) \\
& && (4.15)
\end{aligned}$$

where

$$c_i = \min_{j \in \sigma^{-1}(i)} \left[\frac{d_j - \mathbf{a}_j \cdot \boldsymbol{\mu}}{|\mathbf{b}_j|} \right]. \tag{4.16}$$

The value of I_2 now follows directly from the results in the Appendix:

$$I_2 = \sum_{i=1}^{r} \int_{-\infty}^{c_i} \frac{f(w; 0, 1)dw}{\max_{j \in \sigma^{-1}(i)}(\beta_j w + \gamma_j)} \prod_{k \neq i} \Im[-\infty, \min(c_k, U_{ik}(w))] \tag{4.17}$$

where, letting

$$M_i = \max_{j \in \sigma^{-1}(i)} (\beta_j w_i + \gamma_j),$$

we have that

$$U_{ik} = \min_{j \in \sigma^{-1}(k)} \left[\frac{M_i - \gamma_j}{\beta_j} \right].$$

The value of $\mathbf{I}_1$, which is an r-dimensional vector, can be obtained with a slight modification of the result in the Appendix. Using the same notation as above, for all $l \in [r]$

$$\begin{aligned}
I_1^{(l)} &= \sum_{i=1, i \neq l}^{r} \left\{ \int_{-\infty}^{c_i} \frac{f(w; 0, 1)dw}{\max_{j \in \sigma^{-1}(i)}(\beta_j w + \gamma_j)} \Im_x[-\infty, \min(c_l, U_{il}(w))] \prod_{k \neq i, l} \Im[-\infty, \min(c_k, U_{ik}(w))] \right\} \\
&+ \int_{-\infty}^{c_l} \frac{f(w; 0, 1)wdw}{\max_{j \in \sigma^{-1}(l)}(\beta_j w + \gamma_j)} \prod_{k \neq l} \Im[-\infty, \min(c_k, U_{lk}(w))]
\end{aligned}$$

The integrals $\mathbf{I}_1^{(l)}$ and I_2 are sums of $r < m$ univariate integrals, which can be readily solved using numerical integration techniques such as quadratures or even Simpson approximations.

5 The Local u-Policy

The $\mathbf{u}$- and $\hat{\mathbf{x}}$-policies prescribe a demand-production mapping that is manifestly irrational in some cases. In particular, the problem of "intercomponent" effects arises where a component that gates production for one product will diminish production over all. This arises because the previous policies compute a single α for all products.

On an event by event bases, an "iterated policy" represent rational demand-production mappings, but analytic formulas are not available for the expected production over these policies. A compromise is to compute a separate α for each product. This we refer to as the *local u-Policy.*

As before, for product i, let $\mathcal{D}(i)$ denote the components in the BOM of i. Also, as before, for any component j we can define the subspaces $\Omega_j = \{\mathbf{x} : d_j - \mathbf{a}_j \cdot \mathbf{x} > 0\}$. We now define the subspaces Ω^i for each product i as

$$\Omega^i = \cap_{j \in \mathcal{D}(i)} \Omega_j.$$

If $\mathbf{x} \in \Omega^i$ then component allocation is sufficient to meet product demand for the ith product, even though it may not be sufficient to meet demand for other products.

We make the important observation that in computing expected values for functions $h(\mathbf{x})$ that can be separated into a sum,

$$h(\mathbf{x}) = h_1(x_1) + \cdots + h_n(x_n)$$

the appropriate subspace of integration for the *feasible* production for each function h_j is Ω^j.

For any subspace Ω, let $\mathbf{E}_\Omega$ denote the expectation over support Ω. Thus, the expecta-

tion of h can be expressed as

$$\mathbf{E}h = \sum_{i=1}^{n} [\mathbf{E}_{\Omega^i} h_i(x_i) + \mathbf{E}_{\bar{\Omega}^i} h_i(q_i(\mathbf{x})]$$

where $\mathbf{q}(\mathbf{x})$ is any production policy that maps each $x_i \in \bar{\Omega}^i$ into $q_i(\mathbf{x}) \in \Omega^i$.

We define the *local* u-policy $\mathbf{q}$ as

$$q_i(\mathbf{x}) = \begin{cases} x_i & \text{if } \mathbf{x} \in \Omega^i, \\ x_i - \alpha_i(\mathbf{x}, \mathbf{d}) u_i & \text{otherwise,} \end{cases}$$

where

$$\alpha_i(\mathbf{x}, \mathbf{d}) = \max_{j \in \mathcal{D}(i)} \left[\frac{\mathbf{a}_j \cdot \mathbf{x} - d_j}{\mathbf{a}_j \cdot \mathbf{u}} \right].$$

We now show that with the preceding definition, for any k, we have that $d_k - \mathbf{a}_k \cdot \mathbf{q} > 0$. Substituting $\mathbf{q}$ we get

$$d_k - \mathbf{a}_k \cdot \mathbf{q} = d_k - \mathbf{a}_k \cdot \mathbf{x} + \sum_{i=1}^{n} a_{ki} u_i \max_{j \in \mathcal{D}(i)} \left[\frac{\mathbf{a}_j \cdot \mathbf{x} - d_j}{\mathbf{a}_j \cdot \mathbf{u}} \right].$$

But observe that $a_{ki} = 0$ if $k \notin \mathcal{D}(i)$, and therefore

$$\max_{j \in \mathcal{D}(i)} \left[\frac{\mathbf{a}_j \cdot \mathbf{x} - d_j}{\mathbf{a}_j \cdot \mathbf{u}} \right] \geq \frac{\mathbf{a}_k \cdot \mathbf{x} - d_k}{\mathbf{a}_k \cdot \mathbf{u}},$$

proving that $d_k - \mathbf{a}_k \cdot \mathbf{q} > 0$.

6 The orthomax policy

The orthomax production policy prescribes that for any infeasible demand $\mathbf{x} \in \bar{\Omega}$ the production should be given by transforming $\mathbf{x}$ to w-space, finding the nearest point on the production boundary, and transforming the boundary point back to $\mathbf{x}$ space. Using the

notation of the previous sections, this algorithm translates to:

$$\mathbf{q}(\mathbf{x}) = \begin{cases} TR'\mathbf{w}'(\mathbf{w}(\mathbf{x})) + \boldsymbol{\mu} + \mathbf{U}(\mathbf{x}) & \text{where } \mathbf{x} \in \bar{\Omega} \\ \mathbf{x} & \text{otherwise,} \end{cases} \tag{6.1}$$

where $\mathbf{w}(\mathbf{x}) = RT^{-1}(\mathbf{x} - \boldsymbol{\mu})$ and

$$\mathbf{w}'(\mathbf{w}) = \sum_{i=1}^{r} \min(c_i, w_i)\mathbf{e}_i = (\min(c_1, w_1), \min(c_2, w_2), \ldots, \min(c_r, w_r)) . \tag{6.2}$$

$\mathbf{U}(\mathbf{x})$ is the affine map given by the equation

$$\mathbf{U}(\mathbf{x}) = \boldsymbol{\mu} + TR'RT^{-1}(\mathbf{x} - \boldsymbol{\mu}). \tag{6.3}$$

Conceptually, $\mathbf{U}(\mathbf{x})$ is the affine map that gives the component of $\mathbf{x}$ mapped to $\mathbf{0}$ by the affine map $\mathbf{x} \mapsto \mathbf{w}$.

The expected production under the orthomax policy is given by the integral

$$\begin{aligned} \mathbf{I} &= \int_{\Omega \cup \bar{\Omega}} d\mathbf{x} f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) \mathbf{q}(\mathbf{x}) \\ &= \int_{\Omega \cup \bar{\Omega}} d\mathbf{x} \mathbf{x} f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) + \int_{\bar{\Omega}} d\mathbf{x} f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})(\mathbf{q}(\mathbf{x}) - \mathbf{x}) \\ &= \boldsymbol{\mu} + \mathbf{I}_1 \end{aligned}$$

where $\mathbf{I}_1$ is the integral over $\bar{\Omega}$ on the second line of the formula above.

In order to make progress evaluating $\mathbf{I}_1$, we change the region of integration $\bar{\Omega}$ to the complement of the "orthomax region" given by $\Omega' = \{\mathbf{x} |\ \big(RT^{-1}(\mathbf{x} - \boldsymbol{\mu})\big) \cdot \mathbf{e}_i < c_i \quad \forall i \in [r]\}$. At the same time, we redefine $\mathbf{q}(\mathbf{x})$ to be the equal to the non-identity map on $\bar{\Omega}'$, which is another small, but appreciable approximation. With these approximations, we can write

$$\mathbf{I}_1 = \int_{\bar{\Omega}} d\mathbf{x} f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})(\mathbf{q}(\mathbf{x}) - \mathbf{x}) \approx \int_{\bar{\Omega}'} d\mathbf{x} f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})(\mathbf{q}(\mathbf{x}) - \mathbf{x}). \tag{6.4}$$

A closed form solution can be now obtained:

$$\begin{aligned}
\mathbf{I}_1 &= TR' \int_{-\infty}^{c_1 \cdots c_r} d\mathbf{w} f(\mathbf{w}; 0, I_n)(\mathbf{w} - \mathbf{c}) + \int_{\bar{\Omega}'} d\mathbf{x} f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) \mathbf{U}(\mathbf{x}) &\quad (6.5)\\
&= TR' \left((\Im_x(c_1, \infty) - c_1 \Im(c_1, \infty)), \ldots, (\Im_x(c_r, \infty) - c_r \Im(c_r, \infty)) \right) &\quad (6.6)\\
& &\quad (6.7)
\end{aligned}$$

where we made use of the fact that since $\mathbf{U}(\boldsymbol{\mu} + \boldsymbol{\xi}) = -\mathbf{U}(\boldsymbol{\mu} - \boldsymbol{\xi})$ (i.e $\mathbf{U}$ is odd around $\mathbf{x} = \boldsymbol{\mu}$), $f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})$ is even around $\mathbf{x} = \boldsymbol{\mu}$, and $\mathbf{U}(\mathbf{x}) \cdot \mathbf{a}_i = 0 \forall i$, we have

$$\int_{\bar{\Omega}'} d\mathbf{x} f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) \mathbf{U}(\mathbf{x}) = 0. \tag{6.8}$$

7 Appendix

We show how to reduce the following multivariate integral into a sum of univariate integrals that can be readily computed. Let $\alpha(\mathbf{x}) = G(\max_{i \in [n]} [\beta_i x_i + \gamma_i])$, where $G(x)$ is a well-behaved $\Re \to \Re$ function. We desire to compute the integral

$$I = \left(\prod_{i=1}^{n} \int_{-\infty}^{c_i} \right) d\mathbf{x} f(\mathbf{x}; 0, I_n) \alpha(\mathbf{x}). \tag{7.1}$$

Consider the case when $\beta_1 x_1 + \gamma_1$ is maximum. This occurs when for all $j \neq 1$ $\beta_j x_j + \gamma_j \leq \beta_1 x_1 + \gamma_1$, or when

$$x_j \leq \frac{\beta_1 x_1 + \gamma_1 - \gamma_j}{\beta_j}.$$

Define

$$U_{ij}(x) = \frac{\beta_i x + \gamma_i - \gamma_j}{\beta_j}.$$

Thus we get that

$$
\begin{aligned}
I &= \sum_{i=1}^{n} \int_{-\infty}^{c_i} dx_i f(x_i;0,1) G(\beta_i x_i + \gamma_i) \prod_{j \neq i} \int_{-\infty}^{\min(c_j, U_{ij}(x_i))} f(x_j;0,1) dx_j \\
&= \sum_{i=1}^{n} \int_{-\infty}^{c_i} dx f(x;0,1) G(\beta_i x + \gamma_i) \prod_{j \neq i} \Im[-\infty, \min(c_j, U_{ij}(x))]
\end{aligned}
$$

References

[1] Paul Dagum. Decision analysis under a multivariate normal uncertainty distribution. Technical Report TR-02-98, Rapt Technologies Corporation, 324 Ritch Street, San Francisco, CA 94107, November 1998.

[2] Paul Dagum. Product manufacture risk management. Technical Report TR-04-99, Rapt Technologies Corporation, 324 Ritch Street, San Francisco, CA 94107, February 1999.